

RSA

*Cryptographic Hardware for
Embedded Systems
ECE 3170 A*

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Reading Assignment

- Please read chapter 19 of the course textbook by Schneier

Introduction

- The idea of asymmetric crypto first published by Hellman and Diffie in 1976
 - Martin Hellman was a young faculty member in Electrical Engineering at Stanford
 - Whitfield Diffie joined Stanford's AI Lab in 1969 and worked for John McCarthy
 - In 1974, Whitfield Diffie started to work with Martin Hellman
- RSA invented by Ron Rivest, Adi Shamir and Leonard Adelman in 1977
 - Worked together for a year at MIT to develop a one-way function to implement Diffie-Hellman
 - Rivest and Shamir were trained in Computer Science; Adelman in Mathematics
 - Adelman would typically “break” the proposals of Rivest and/or Shamir
- Additional contemporary inventors of asymmetric cryptographic algorithms
 - Ralph Merkle developed an asymmetric crypto scheme as an undergraduate project at U.C. Berkeley and unsuccessfully tried to publish prior to 1977
 - Clifford Cocks, Government Communications Headquarters (GCHQ), U.K., also in the 1970s

RSA Overview

- Security based on difficulty of factoring large numbers
 - Public and private keys are functions of prime numbers
 - E.g., each key may have 200 digits
- Concept of a one-way function
 - Easy to compute in one direction, hard to compute in reverse
- Different keys accomplish encryption versus decryption
 - Allows one to publish encryption keys while keeping decryption keys secret
 - “Public-key” cryptography

Key Generation

- Choose two large prime numbers at random
 - p and q of approximately equal length
- Compute $n = p * q$
- Randomly choose encryption key e
 - e and $(p - 1)(q - 1)$ have no factor in common (other than 1)
 - Note: a number x is said to **factor** y iff x/y has no remainder (i.e., the remainder is zero!)
 - relatively prime or *coprime*
- Compute d such that the following holds
 - $ed = 1 \bmod (p - 1)(q - 1)$
 - In other words, $d = e^{-1} \bmod (p - 1)(q - 1)$
 - Note that in modular arithmetic e^{-1} is the multiplicative inverse of e

A Comment on Multiplicative Inverse in Modular Arithmetic

- In modular arithmetic, all numbers are whole
 - No fractions and no decimals
- For example, consider modulus with respect to u
- Suppose $u = 11$
 - $13 \bmod u = 13 \bmod 11 = 2$
- Suppose $t = 7$
 - Let $v = t^{-1}$ = multiplicative inverse of t in modular arithmetic
 - $tv \bmod u = 1 = t * t^{-1}$
- Compute v such that $v = t^{-1}$
 - try $v = 1$: then $tv \bmod u = 7 \bmod 11 = 7 \neq 1$
 - try $v = 2$: then $tv \bmod u = 14 \bmod 11 = 3 \neq 1$
 - ...
 - try $v = 8$: then $tv \bmod u = 56 \bmod 11 = 1$
 - $v = t^{-1} = 8$

Key Generation (cont'd)

- p and q are randomly chosen large prime numbers
- $n = p * q$
- Encryption key e is chosen such that
 - e and $(p - 1)(q - 1)$ have only 1 as a factor in common
 - relatively prime or *coprime*
- Compute d such that the following holds
 - $d = e^{-1} \bmod (p - 1)(q - 1)$ where e^{-1} is the multiplicative inverse of e
- d and n have no factor in common (other than 1)
 - d and n are coprime
- e and n are the public “key”
- d and n are the private “key”

Example of Key Generation

- Choose $p = 47, q = 71$
 - $n = p * q = 47 * 71 = 3337$
- e must have no factor in common with $(p - 1)(q - 1) = 46 * 70 = 3220$
 - Randomly choose $e = 79$
 - $d = e^{-1} \bmod (p - 1)(q - 1) = 79^{-1} \bmod 3220$
 - a program can be run to find d
 - variations of an original approach by Euclid
 - the result for this example is $d = 1019$
 - $ed = 79 * 1019 = 80501$
 - $3220 * 25 = 80500$
 - $ed \bmod 3220 = 80501 \bmod 3220 = 1$
- Public key (pair): $e = 79$ and $n = 3337$
- Private key: $d = 1019$
- Note: p and q may be thrown away & are no longer used or needed
 - Of course, p and q should not be revealed in any way!

RSA Encryption and Decryption

- Given message m , divide m into blocks m_i each of size 2^s
 - E.g., s is typically the largest number such that $2^s < n$
- The encrypted message c has blocks c_i each equal in size to m_i
- Encryption formula: $c_i = m_i^e \bmod n$
- Decryption formula: $m_i = c_i^d \bmod n$
 - Note the following where we skip the “mod n ” everywhere:
 - $c_i^d = (m_i^e)^d = m_i^{ed} = m_i^{k(p-1)(q-1) + 1}$ since $ed = 1 \bmod (p-1)(q-1)$
 - $m_i^{k(p-1)(q-1) + 1} = m_i m_i^{k(p-1)(q-1)} = m_i (m_i^{(p-1)(q-1)})^k = m_i * 1 = m_i$
 - Note that $(m_i^{(p-1)(q-1)})^k \bmod n = 1$ due to known results in group theory
- Note that may also encrypt with d and decrypt with e

Example of Encryption with Keys from Prior Ex.

- Public key (pair): $e = 79$ and $n = 3337$
- Private key (pair): $d = 1019$ and $n = 3337$
- $m = 6882326879666683$,
 - $m_1 = 688$
 - $m_2 = 232$
 - $m_3 = 687$
 - $m_4 = 966$
 - $m_5 = 668$
 - $m_6 = 003$
- $c_1 = 688^{79} \bmod 3337 = 1570$, $c = 1570\ 2756\ 2091\ 2276\ 2423\ 158$
- $m_1 = 1570^{1019} \bmod 3337 = 688$, and so on