# HOST 2026 Tutorial:
# Part I
# Lightweight Cryptography and Mathematics Overview
# May 4, 2026

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# Acknowledgement


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# Outline

- Introduction
- Research Overview
- Lightweight Cryptography
- Background
  - Terminology
  - Feedback Registers
  - Finite Field Theory
- Chaining Period Theorem
- Product Registers
  - Mersenne Product Registers
  - Composite Mersenne Product Registers (CMPRs)
  - CMPR Theorems
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# Introduction

- Arman Allahverdi
  - B.S. in Electrical Engineering from the University of New Mexico, Albuquerque, New Mexico, 2021
  - M.S. in Electrical and Computer Engineering from The Georgia Institute of Technology, Atlanta, Georgia, 2024
  - 4th-year Ph.D. student in Electrical and Computer Engineering (Hardware-Software Codesign for Security) at The Georgia Institute of Technology

# Introduction

- Vincent John Mooney III
  - B.S. in Electrical Engineering, Yale University, 1991
  - B.S. in Computer Science, Yale University, 1991
  - M.S. In Electrical Engineering, Stanford University, 1994
  - M.A. in Philosophy, 1997
  - Ph.D. in Electrical Engineering, 1998
  - Associate Professor, School of Electrical and Computer Engineering, The Georgia Institute of Technology
  - Adjunct Associate Professor, School of Computer Science, The Georgia Institute of Technology

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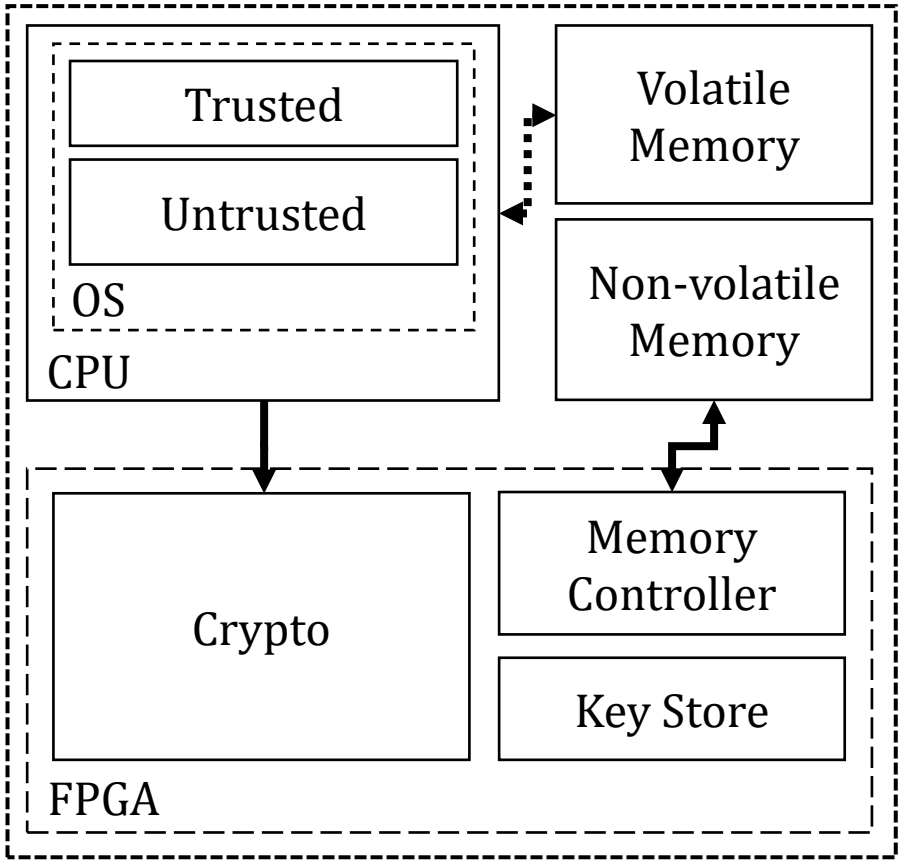
# Research Overview

- With the advent of battery powered and ultra-small devices requiring security, **lightweight cryptography has become an area of research interest**, for example with the proposal of hardware-oriented cryptographic schemes in the NIST Lightweight Cryptography Competition (2018-2023)
- Nonlinear Feedback Shift Registers (NLFSRs) are a valuable building block for lightweight cryptographic hardware, as they generate periodic nonlinear state sequences and can be efficiently implemented in hardware
  - NLFSRs do not automatically scale with register size, and so as a result there is no efficient method of building an NLFSR of an arbitrary size

# Research Impact

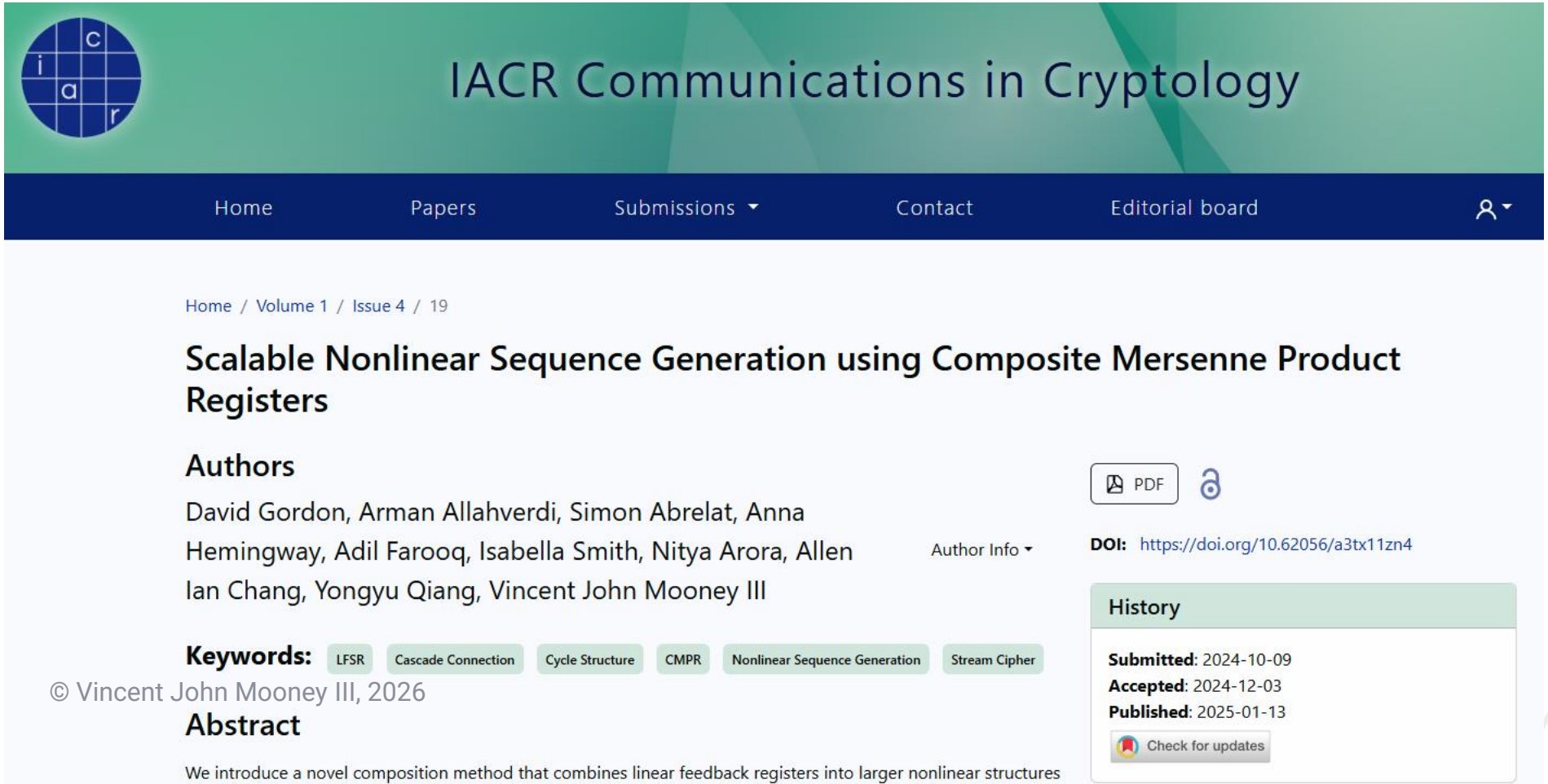
- We have discovered a new, scalable class of nonlinear feedback register that can outperform the prior state-of-the-art in lightweight cryptography
  - Scalable to large values (e.g., 256, 1024 and larger) based on Mersenne primes
  - Mathematical proof of exponential expected period
  - Hardware-efficient due to register-based construction
  - Exponential number of hardware constructions allow for the key to be synthesized into the register structure, thus producing very low area in reconfigurable logic compared to other lightweight options




- In summary, CMPRs provide the first nonlinear feedback structure scalable to thousands of bits of size with provable exponential state and a similarly exponential number of distinct CMPR structures

# IACR Communications in Cryptology Publication

[1] D. Gordon, A. Allahverdi, S. Abrelat, A. Hemingway, A. Farooq, I. Smith, N. Arora, A. Chang, Y. Qiang and V. Mooney, "Scalable Nonlinear Sequence Generation using Composite Mersenne Product Registers," IACR Communications in Cryptology, 1(4), pp. 1-77, January 2025, https://doi.org/10.62056/a3tx11zn4.



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**Scalable Nonlinear Sequence Generation using Composite Mersenne Product Registers**

**Authors**

David Gordon, Arman Allahverdi, Simon Abrelat, Anna Hemingway, Adil Farooq, Isabella Smith, Nitya Arora, Allen Ian Chang, Yongyu Qiang, Vincent John Mooney III

Author Info

PDF

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Check for updates

**Abstract**

We introduce a novel composition method that combines linear feedback registers into larger nonlinear structures





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# Lightweight Cryptography

- **NIST Lightweight Cryptography Workshops:** 2015-2023
  - **Goal:** Get feedback on industry need, relevant applications, and security requirements
- **NIST Lightweight Cryptography Competition (LWC):** 2018-2023 [3]
  - **Goal:** Source secure, hardware-efficient symmetric-key algorithms from the cryptography community
  - **Call for Algorithms:** 2018
  - **Scope of Algorithms:** encryption, authentication, hashing
  - Candidate algorithms selected in several phases
  - Evaluation process concerned with (i) hardware + software efficiency, (ii) security analysis, and (iii) overall flexibility and additional features (e.g., different modes of operation)
  - Ascon [4] selected as winner out of 10 finalists

# Lightweight Cryptography

- **NIST Lightweight Cryptography Competition:** 2018-2023 [3]
  - Candidate algorithms required to support authenticated encryption with associated data (AEAD)
  - Minimum key size of 128 bits required
    - At least $\approx 2^{128}$ security against key recovery attacks
  - Candidate algorithms proposing a hash function were required to have at least $2^{112}$ security against a classical computer

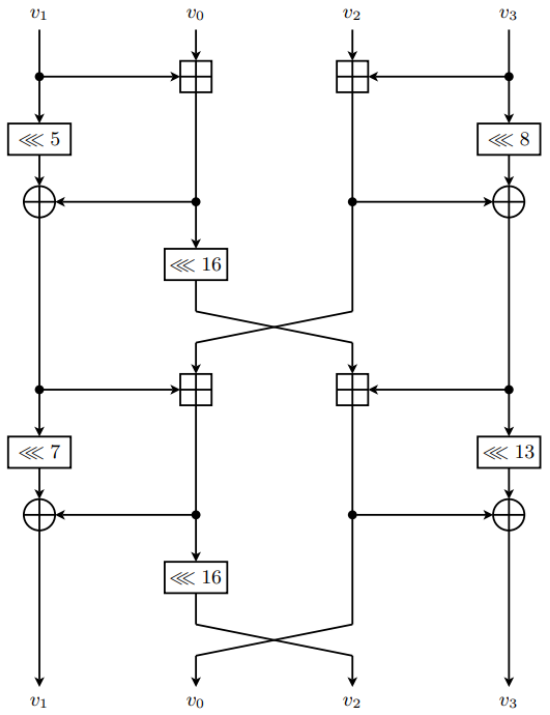
| AEAD-only | AEAD + Hashing |
|---|---|
| Grain-128AEADv2 | Ascon |
| GIFT-COFB | PHOTON-Beetle |
| ISAP | SPARKLE |
| TinyJAMBU | Xoodyak |
| Elephant | Romulus |

**Cryptographic Capabilities of the 10 NIST LWC Finalists**

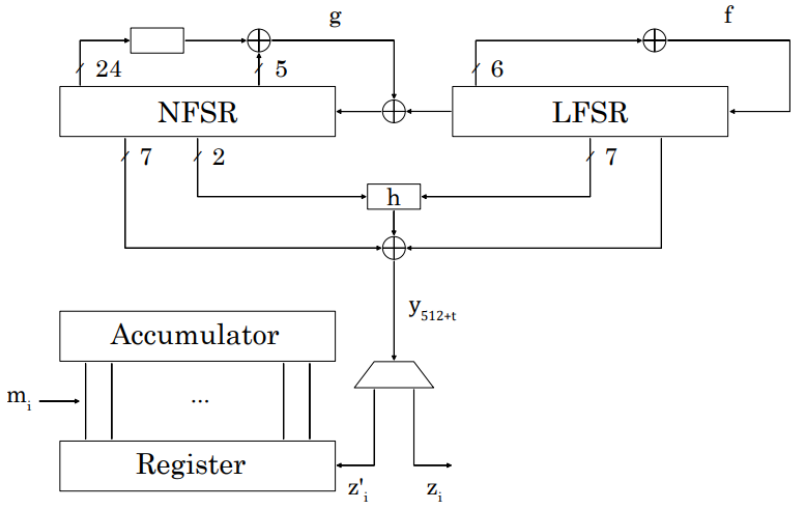
 

# Lightweight Cryptography

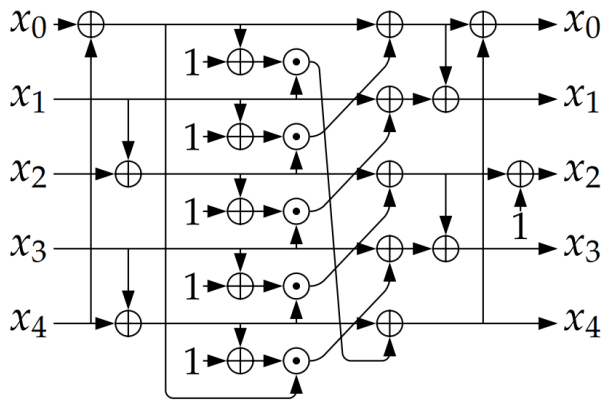
- Different hardware paradigms for lightweight cryptosystem design:
  - ARX (Add-Rotate-XOR)
  - Register-based (LFSRs, NLFSRs, and this work)
  - S-Boxes implemented as Boolean circuits (ex: Ascon)




Chaskey [5] (ARX)




Grain-128AEADv2 [6] (Register-Based)




Ascon [4] S-Box





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# Terminology

- **Mersenne Exponent:** An integer $n$ such that $2^n - 1$ is prime
- **Mersenne Prime:** A prime number in the form $2^n - 1$
- **Feedback Register:** A form of register where the next state, $A[t+1]$, is governed by a function $f$ of the current state $A[t]$ (e.g., $A[t+1] = f(A[t])$
- **Feedback Shift Register:** A form of feedback register where the output of each state bit $a_i$ is connected to the input of the next bit $a_{i+1}$
- **Linear Feedback Shift Register (LFSR):** A feedback shift register where successive states are determined by a linear function of the current state
- **Nonlinear Feedback Shift Register (NLFSR):** A feedback shift register where successive states are determined by a nonlinear function of the current state
- **Feedback Polynomial:** A polynomial that determines the feedback taps of a feedback register
- **Primitive Polynomial:** An irreducible (unfactorable) polynomial of degree $n$ over a finite field $GF(p)$ whose roots are primitive elements of the extension field $GF(p^m)$

# Terminology (continued)

- **Product Register:** A feedback register that performs Galois Field Multiplication of the current state $A[t]$ and an update polynomial $U$, modulo a feedback polynomial $P$
- **Mersenne Product Register (MPR):** A product register of size $n$, where $n$ is a Mersenne exponent
- **Composite Mersenne Product Register (CMPR):** A feedback register formed by several MPRs combined through nonlinear chaining functions
- **Stream Cipher:** An encryption scheme that encrypts data by performing an XOR with a pseudorandom bitstream
- **Keystream:** A pseudorandom bitstream generated from a secret key and a public initialization vector (IV)
- **Chaining Density:** The proportion of the bits of a CMPR that receive chaining functions
- **Initialization Round:** A single clocking of a feedback register (application of the next state function to the feedback register)
- **Z-transform:** A mathematical transformation of a sequence in discrete time into the frequency domain, i.e., a complex-valued representation of the sequence [13]

# Mersenne Examples

- 13 is a Mersenne exponent because $2^{13} - 1 = 8191$ which is a prime number
- 13 is not a Mersenne prime because there is no integer $n$ such that $2^n - 1 = 13$
- 11 is a prime number that is neither a Mersenne exponent nor a Mersenne prime
- 7 is both a Mersenne exponent and a Mersenne prime
- The first 20 Mersenne exponents are 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253 and 9689
- 8191 is a Mersenne prime (with Mersenne exponent 13) but is not a Mersenne exponent
- All Mersenne exponents are prime numbers

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# Feedback Register Definition and Example

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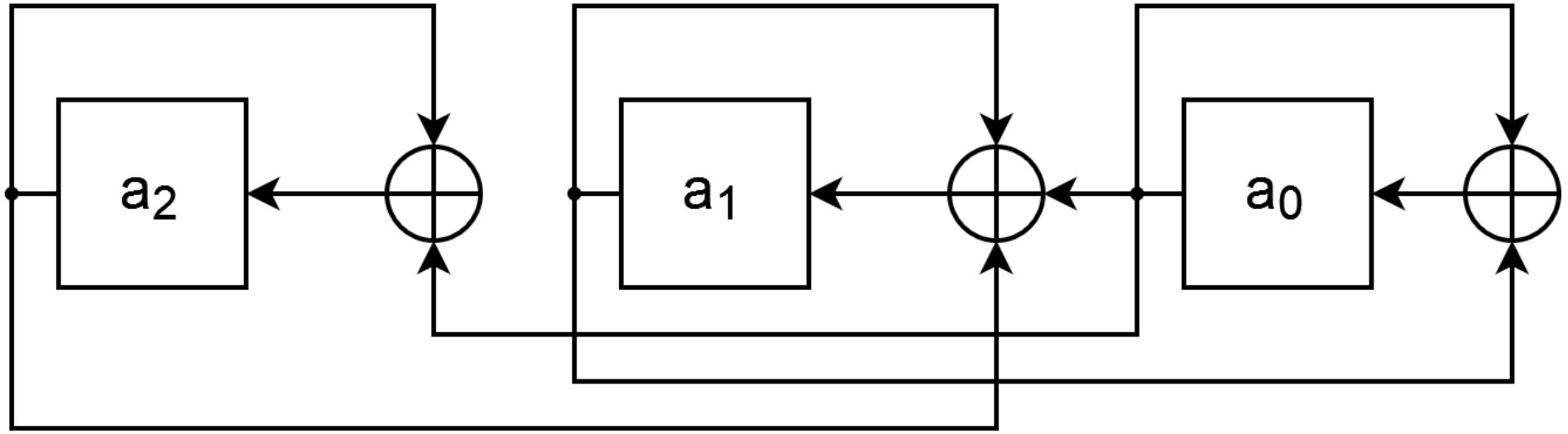
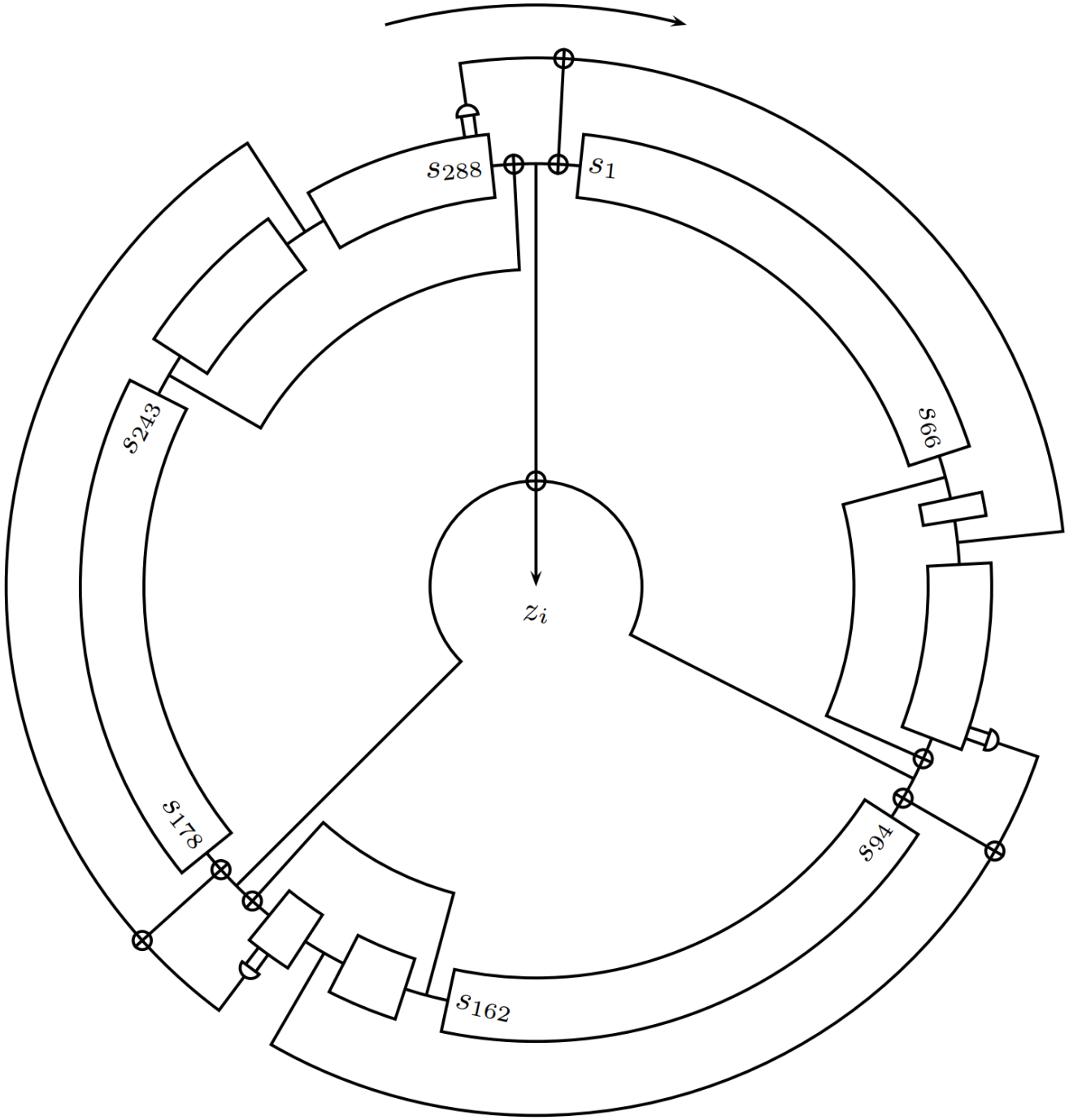


# Feedback Shift Register Definition and Example

- **Feedback Shift Register:** A form of feedback register where the output of each state bit $a_i$ is connected to the input of the next bit $a_{i+1}$
- **Nonlinear Feedback Shift Register (NLFSR):** A feedback shift register where successive states are determined by a nonlinear function of the current state



# Linear Feedback Shift Register Definition and Examples

- **Linear Feedback Shift Register (LFSR):** A feedback shift register where successive states are determined by a linear function of the current state
- **Feedback Polynomial:** A polynomial that determines the feedback taps of a feedback register

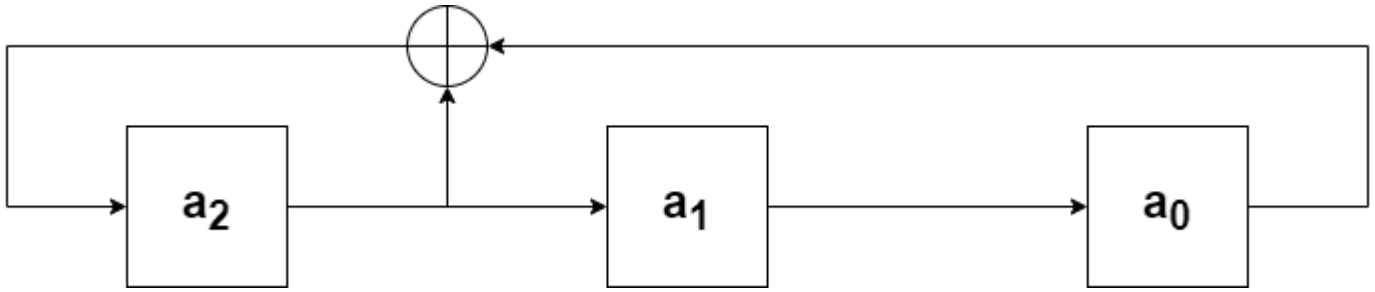



**Figure 1(a): LFSR in Fibonnaci Configuration [1]**

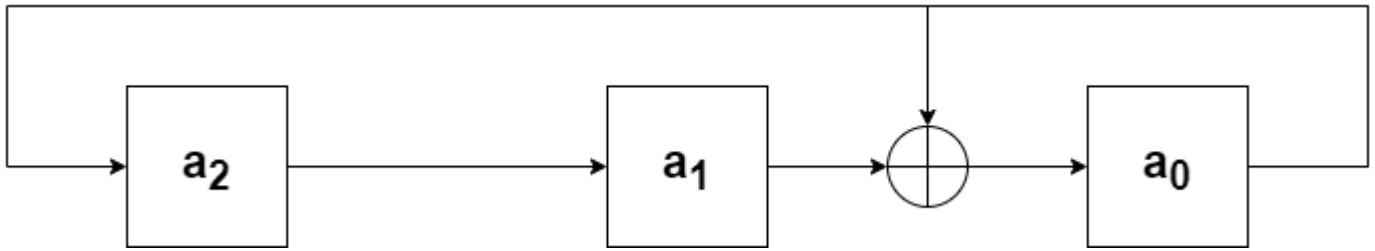
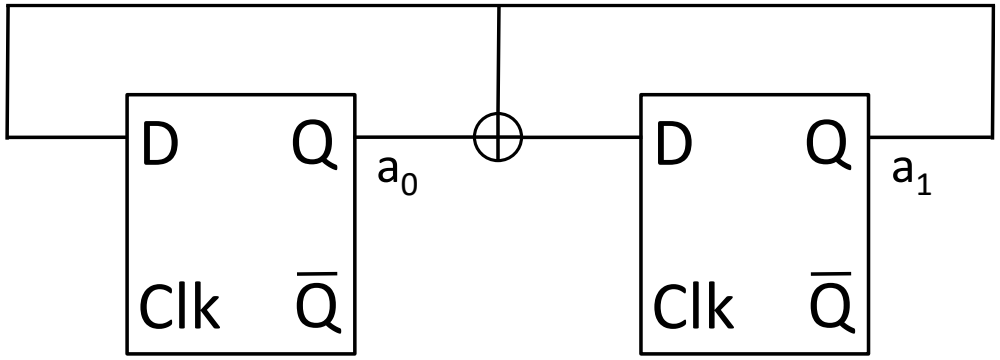


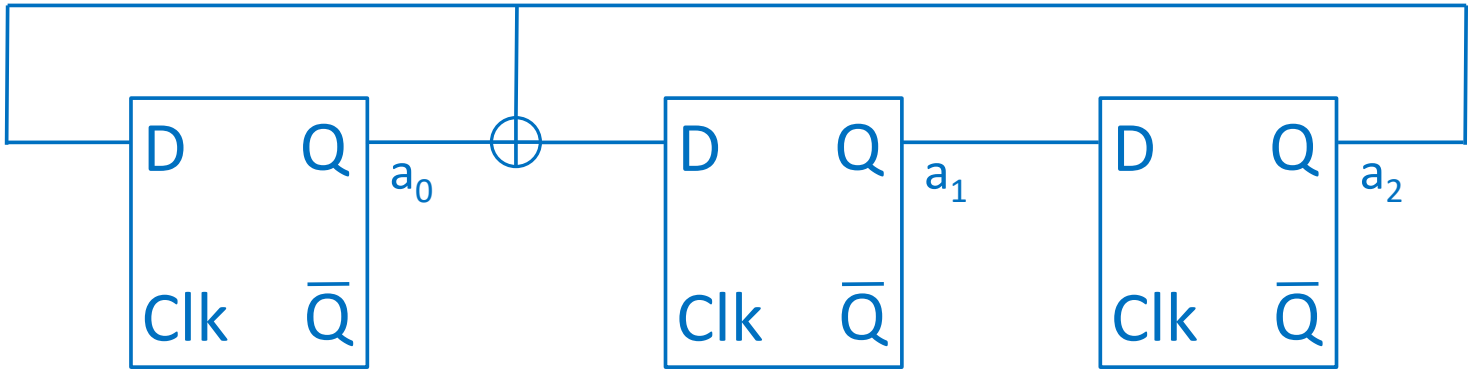

**Figure 1(b): LFSR in Galois Configuration [1]**

# Linear Feedback Shift Register Examples

- **Primitive Polynomial:** An irreducible (unfactorable) polynomial of degree $n$ over a finite field $GF(p)$ whose roots are primitive elements of the extension field $GF(p^m)$
- LFSR, P=$x^2$+x+1, two cycles: {01,11,10} and {00}




- LFSR, P= $x^3$+x+1, two cycles: {000} and {001,010,100,011,110,111,101}



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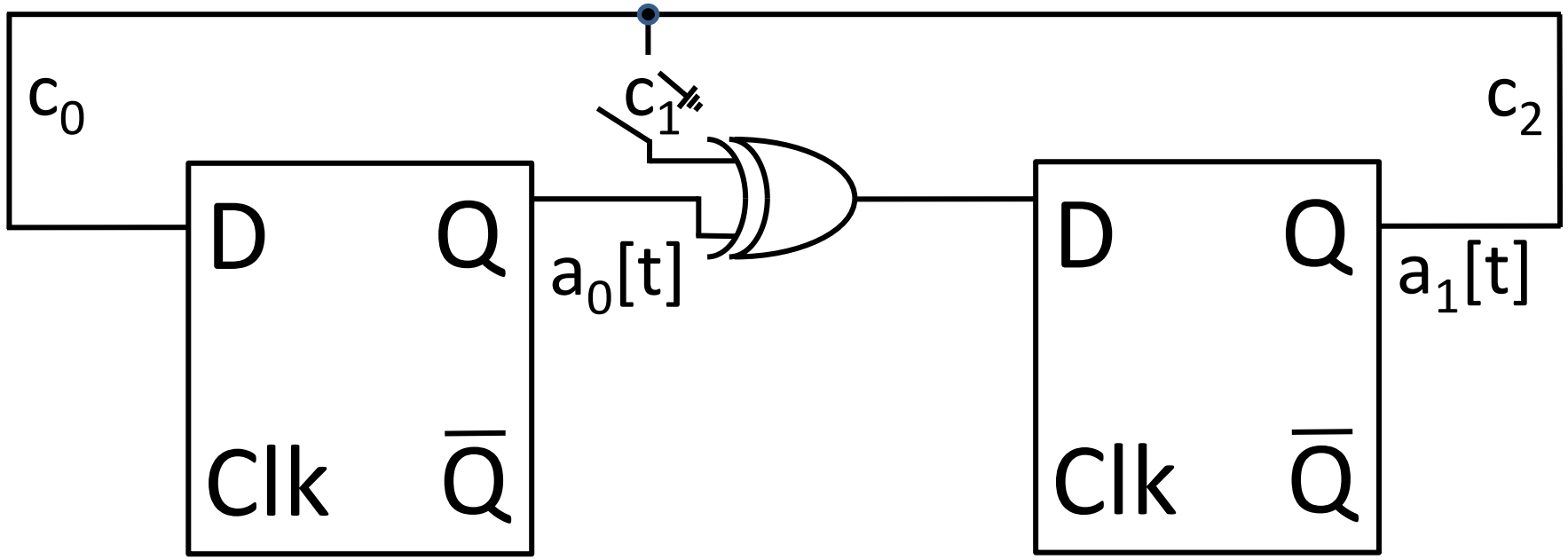
# Polynomial Representation

- Galois tried to find the roots of the quintic equation $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ using the coefficients for a general formula, similar to $ax^2 + bx + c$ where the quadratic formula is expressed in terms of *a*, *b* and *c*
- Can view the bits in a feedback shift register as coefficients in a polynomial equation where $x^5$, $x^4$, $x^3$, $x^2$, $x^1$, $x^0$, etc., are placeholders (i.e., not evaluated or substituted for with numbers)
- Multiplication by x, modulus the characteristic polynomial, calculates the next state
  - $f(x)[t] = \sum_{i=0}^{n-1} a_i[t]x^i$
  - $P(x) = \sum_{i=0}^{n} c_i x^i$
  - $f(x)[t+1] = xf(x)[t] \bmod P(x)$

# Example Computation in GF($2^n$)

- Can be quickly implemented in hardware with feedback shift registers




- $f(x)[t] = a_1[t]x + a_0[t]$ , $P(x) = c_2x^2 + c_1x + c_0$ , $f(x)[t+1] = xf(x)[t] \bmod P(x)$

# Feedback Registers, Field Theory and Periodicity

- A feedback register is a register $A$ such that on each clock cycle (state update), the register state $A[t]$ is updated according to some function $f$, such that:
  - $\boldsymbol{A[t+1] = f(A[t])}$
- In the prior work, feedback registers show up in many different forms:
  - **LFSRs:** $f$ is a linear function of $A[t]$, **can be used as a component in lightweight cryptography provided nonlinear components are also present**
    - **Two Possible Configurations:** Galois ("Internal-XOR"), Fibonacci ("External-XOR")
    - This presentation focuses on the Galois configuration
  - **NLFSRs:** $f$ is a nonlinear function of $A[t]$, **can be used as a standalone component in lightweight cryptography**
  - **De Bruijn Sequences, T-Functions, Cross-Joined Pairs [1]:** Derived from LFSR and NLFSR theory, imposing special restrictions on $f$
- LFSRs and NLFSRs are parametrized by their feedback polynomials $P(x)$
- For a Galois LFSR, the state updates according to:
  - $\boldsymbol{A[t+1] = xA[t] \bmod P(x)}$

# Background: Feedback Registers and Periodicity

- Generally, an $n$-bit feedback register can hold $2^n$ possible unique states
- The **period** of a feedback register refers to the number of unique states the register assumes before returning to its starting state, but it is not always possible to reach a period of $2^n$
- Most LFSRs and NLFSRs have a maximum attainable period of $2^n - 1$ for an $n$-bit register size
  - Maximum/full-period LFSRs can be constructed by ensuring $P(x)$ is a primitive polynomial
  - Full-period NLFSRs exist for small register sizes [7]
  - **No scalable methodology or mathematical process for constructing full-period NLFSRs of arbitrary size, with limited exceptions:**
    - For example, cross-joined pairs [8] can be used to construct larger full-period NLFSRs; however, such NLFSRs have limitations: at most half of the state updates are nonlinear, and the feedback polynomial must depend only on linearly-updated state variables

# Background: Feedback Register-Based LWC

| Design | Capabilities | Components | Security |
|---|---|---|---|
| TRIVIUM [9] | Encryption | NLFSR | $2^{80}$ (Key) |
| Espresso [10] | Encryption | NLFSR (Cross-Joined) | $2^{128}$ (Key) |
| Grain-128AEADv2 [6] | AEAD | LFSR + NLFSR | $2^{128}$ (Key)<br>$2^{64}$ (Tag) |
| Quark [11] | Hashing | LFSR + NLFSRs | Varies<br>$2^{64}$ up to $2^{112}$<br>(Collision Resistance) |

- Cryptographic primitives based on LFSRs and NLFSRs have lightweight hardware implementations
- Limitations:
  - NLFSRs require many clock cycles for the internal state to appear randomized
  - Scaling NLFSRs to support different security levels is not straightforward

# Terminology (cont'd)

- **Product Register:** A feedback register that performs Galois Field Multiplication of the current state $A[t]$ and an update polynomial $U(x)$, modulo a feedback polynomial $P(x)$:
  - $A[t+1] = U(x)A[t] \; mod \; P(x)$

256-bit Product Register

$$U(x) = x^2$$

$$P(x) = x^{256} + x^{213} + x^{133} + x^{115} + x^{48} + x^{12} + 1$$

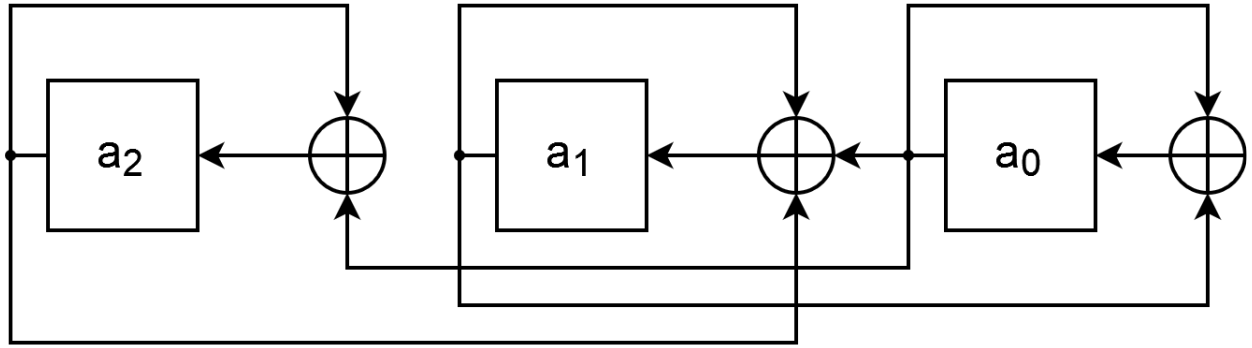
**An Example of a 256-bit PR with a Primitive Feedback Polynomial**

**State Update:**

$$A[t+1] = x^2\left(a_{255}x^{255} + \cdots + a_o\right) \; mod \; P(x)$$

# Terminology (cont'd)

- **Mersenne Product Register (MPR):** A product register of size $n$, where $n$ is a Mersenne exponent




- **Composite Mersenne Product Register (CMPR):** A feedback register formed by several MPRs combined through nonlinear chaining functions

# Terminology (cont'd)

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# Definitions

- **Definition 1:** A discrete-time finite-state dynamical system is describe by the pair $(S, f)$ where $S$ is a finite set of states for the system and $f: S \to S$ is a function which describes how the system evolves at each time step; we further denote by  the state of the system at time $t$, with the initial state of the system being at time $t = 0$

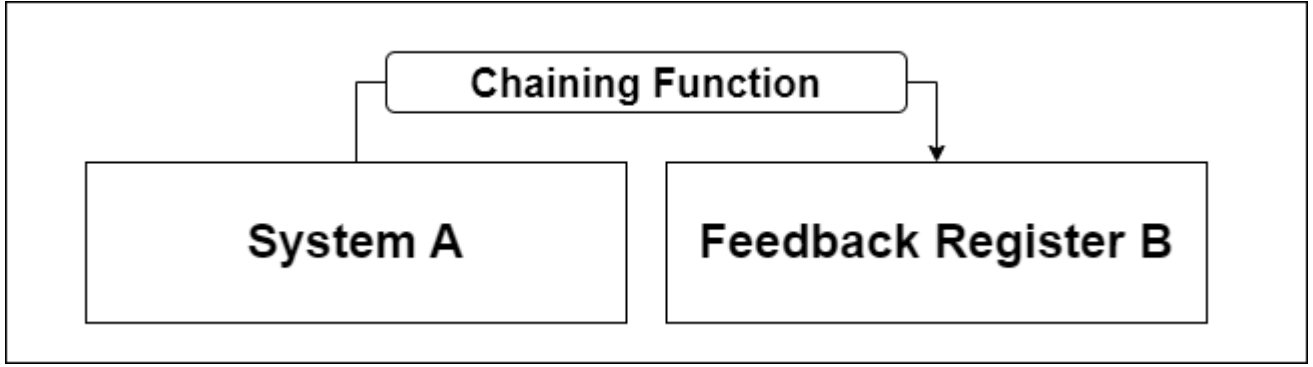



Figure 2(a) from [1]

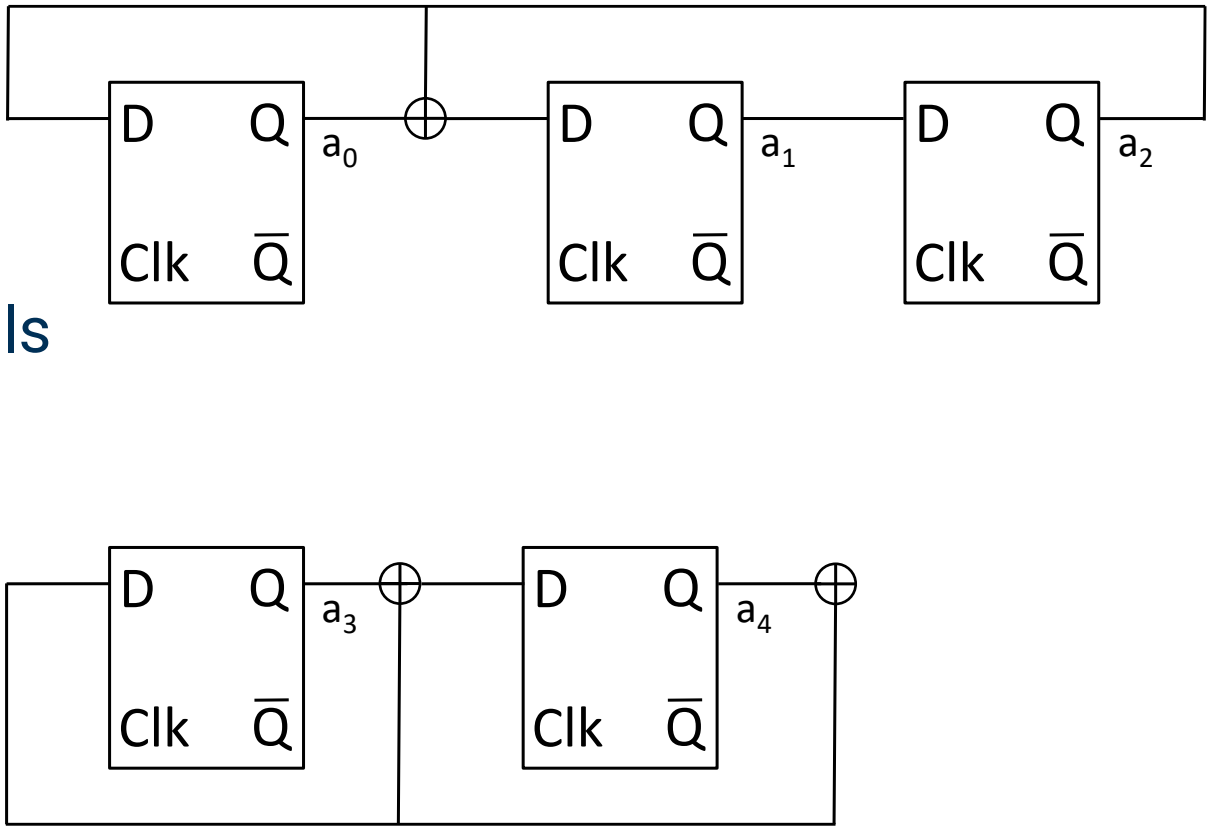
***THEOREM 1 (Chaining Period Theorem)*** *As shown in Figure 2(a), let A=(S,f) be a nonsingular system, and let B be an n-bit register with linear feedback represented by the matrix* $U \epsilon \mathbb{GL}(n,2)$*. Let the characteristic polynomial of U be irreducible, and let B have period m. Let* $C: S \rightarrow \mathbb{F}_2^n$ *be some chaining function from A to B, and let R denote the composite system formed by A and B via chaining function C. Then the cycle structure of R can be determined from the cycles of A and B as follows:*

*For any cycle of length p in A:*

1. *If* $m \mid p$ *and* $f_p = 1_{G_U}$ *then there are* $2^n$ *cycles of length p*
2. *If* $m \mid p$ *and* $f_p \neq 1_{G_U}$ *then there are* $2^{n-1}$ *cycles of length 2p*
3. *If* $m \nmid p$ *then there are* $\frac{2^n-1}{m}\gcd(m,p)$ *cycles of length* $\mathrm{lcm}(m,p)$ *and one cycle of length p*

# Example 1

- Shown is a 3-bit LFSR and a 2-bit LFSR
- These two LFSRs with primitive polynomials were shown earlier
- The 2-bit LFSR is relabeled $a_3$, $a_4$



# Example 1

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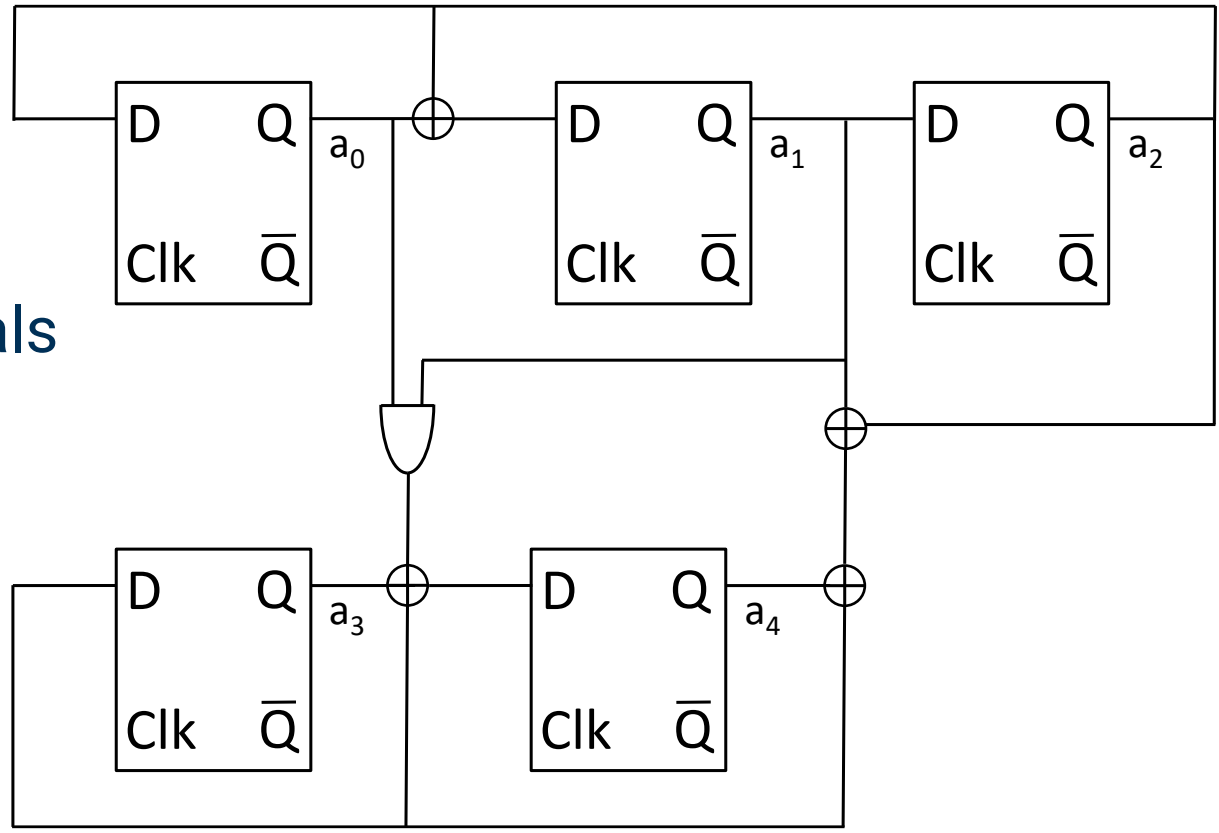
- These two LFSRs with primitive polynomials were shown earlier
- The 2-bit LFSR is relabeled $a_4$, $a_5$
- The chaining function is $a_1$ XOR $a_2$ and $a_0$ AND $a_1$
- The next state functions for the 2-bit LFSR are now $a_4 = a_3$ XOR ($a_0$ AND $a_1$) and $a_3 = a_4$ XOR $a_1$ XOR $a_2$



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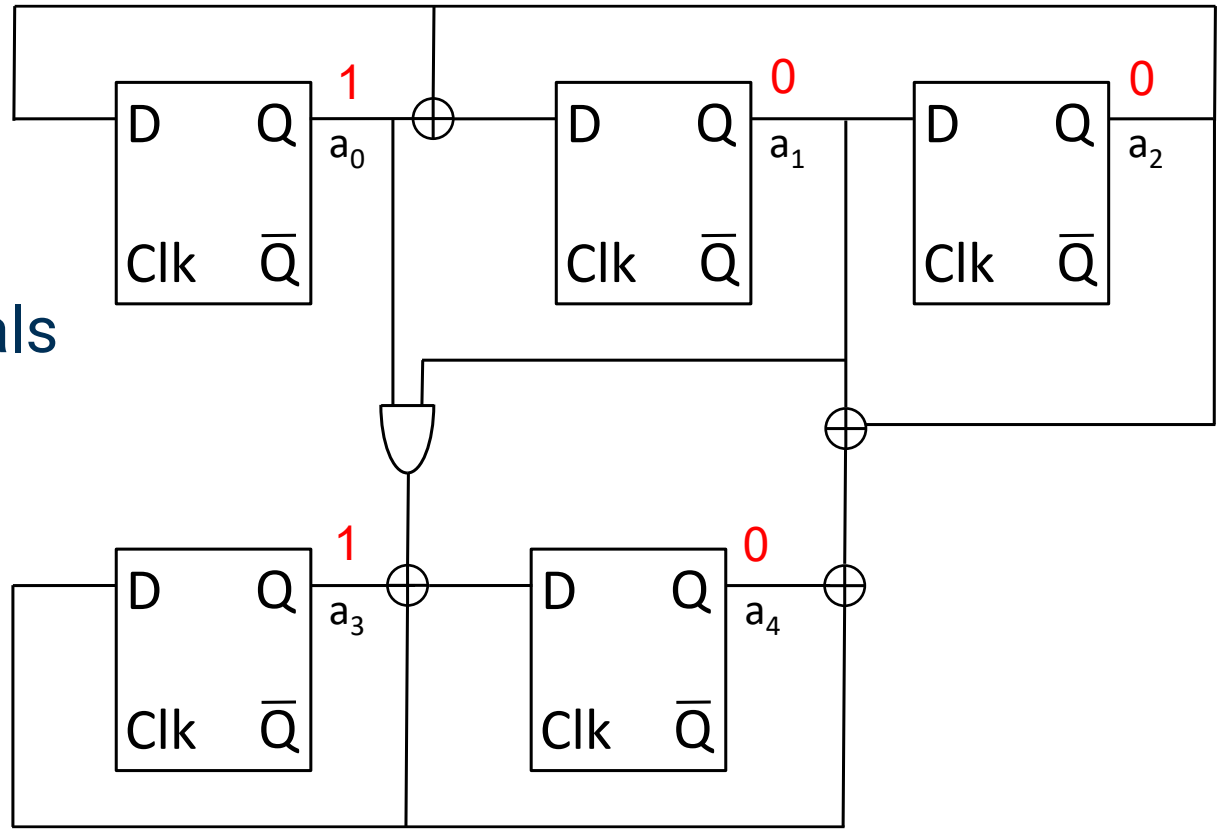


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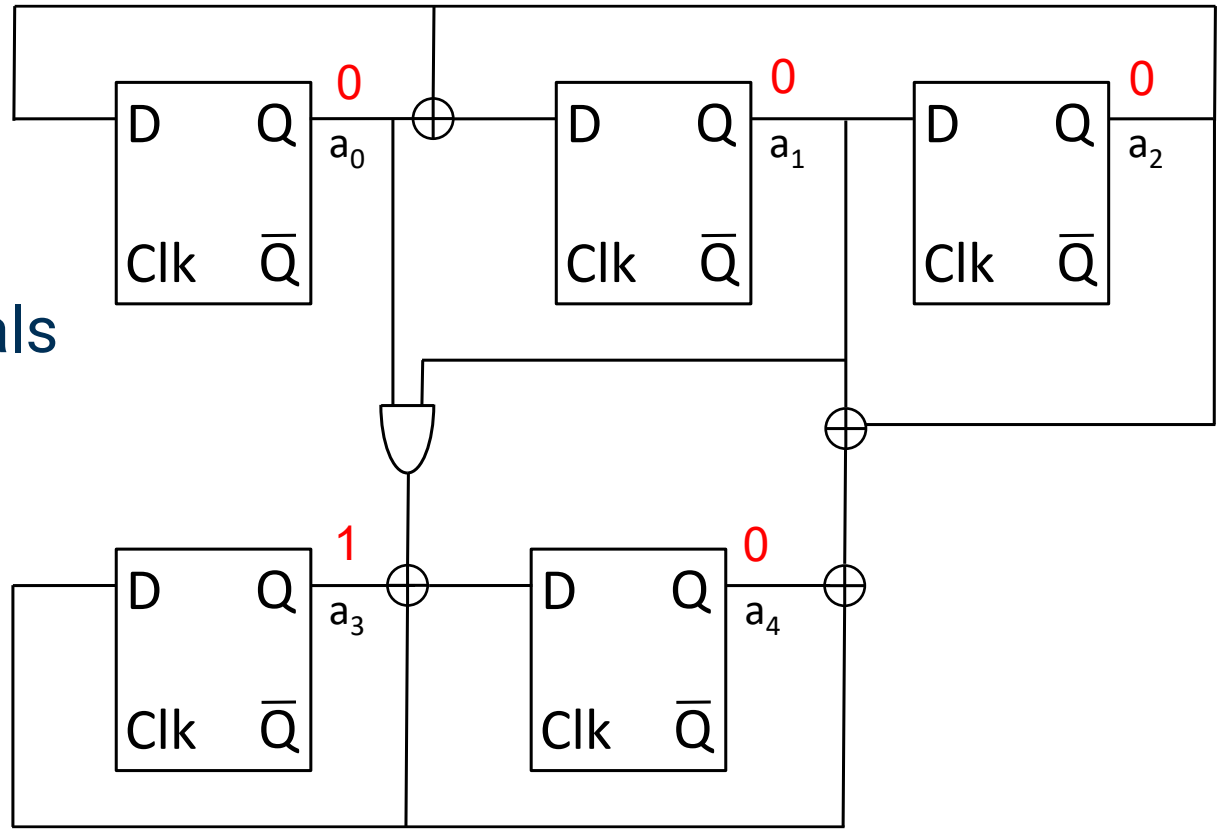
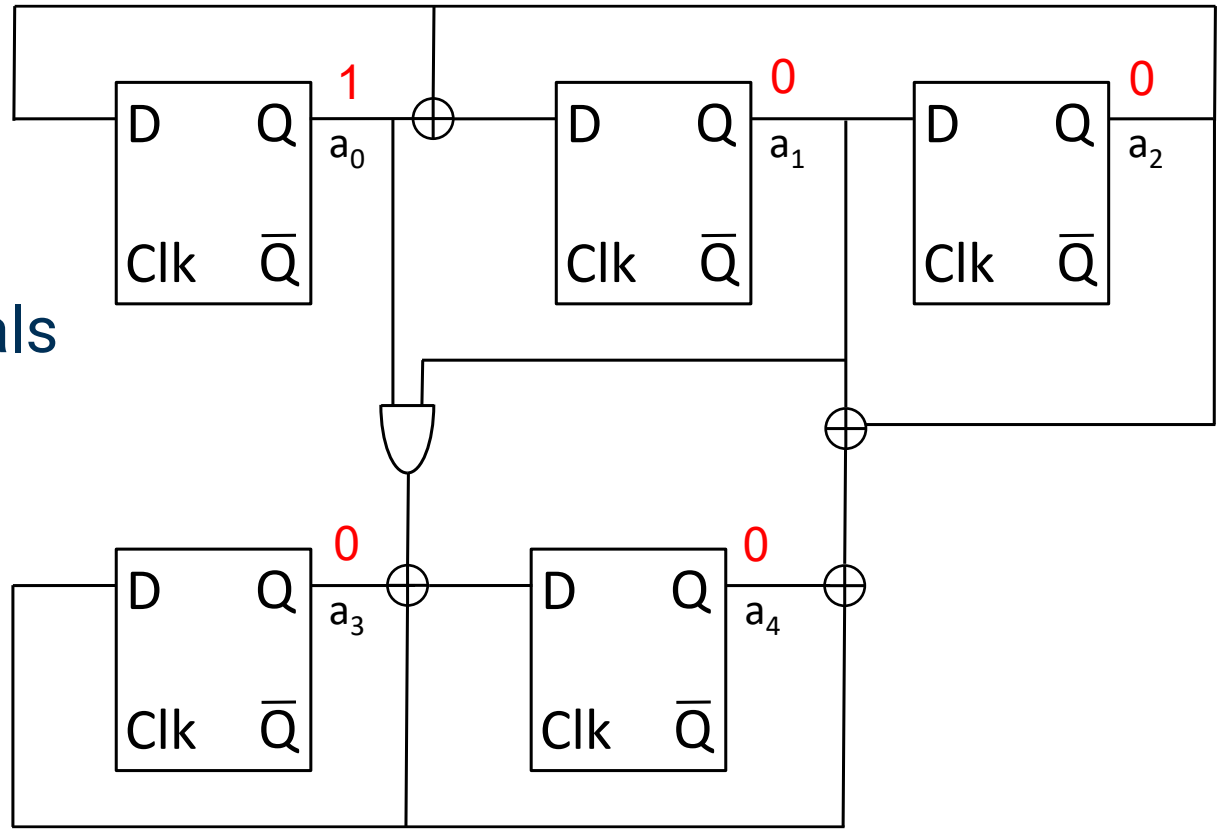
- The 2-bit LFSR is relabeled $a_4$, $a_5$
- The chaining function is $a_1$ XOR $a_2$ and $a_0$ AND $a_1$
- The next state functions for the 2-bit LFSR are now $a_4 = a_3$ XOR ($a_0$ AND $a_1$) and $a_3 = a_4$ XOR $a_1$ XOR $a_2$
- Consider starting state $a_2,a_1,a_0 = 001$ and $a_4,a_3 = 01$; the result is a cycle of size 21

# Example 1




- Shown is a 3-bit LFSR and a 2-bit LFSR
- These two LFSRs with primitive polynomials were shown earlier
- The 2-bit LFSR is relabeled $a_4$, $a_5$
- The chaining function is $a_1$ XOR $a_2$ and $a_0$ AND $a_1$
- The next state functions for the 2-bit LFSR are now $a_4 = a_3$ XOR ($a_0$ AND $a_1$) and $a_3 = a_4$ XOR $a_1$ XOR $a_2$
- Consider starting state $a_2,a_1,a_0 = 001$ and $a_4,a_3 = 01$; the result is a cycle of size 21
- Consider starting state $a_2,a_1,a_0 = 000$ and $a_4,a_3 = 01$; the result is a cycle of size 3

# Example 1




- Shown is a 3-bit LFSR and a 2-bit LFSR
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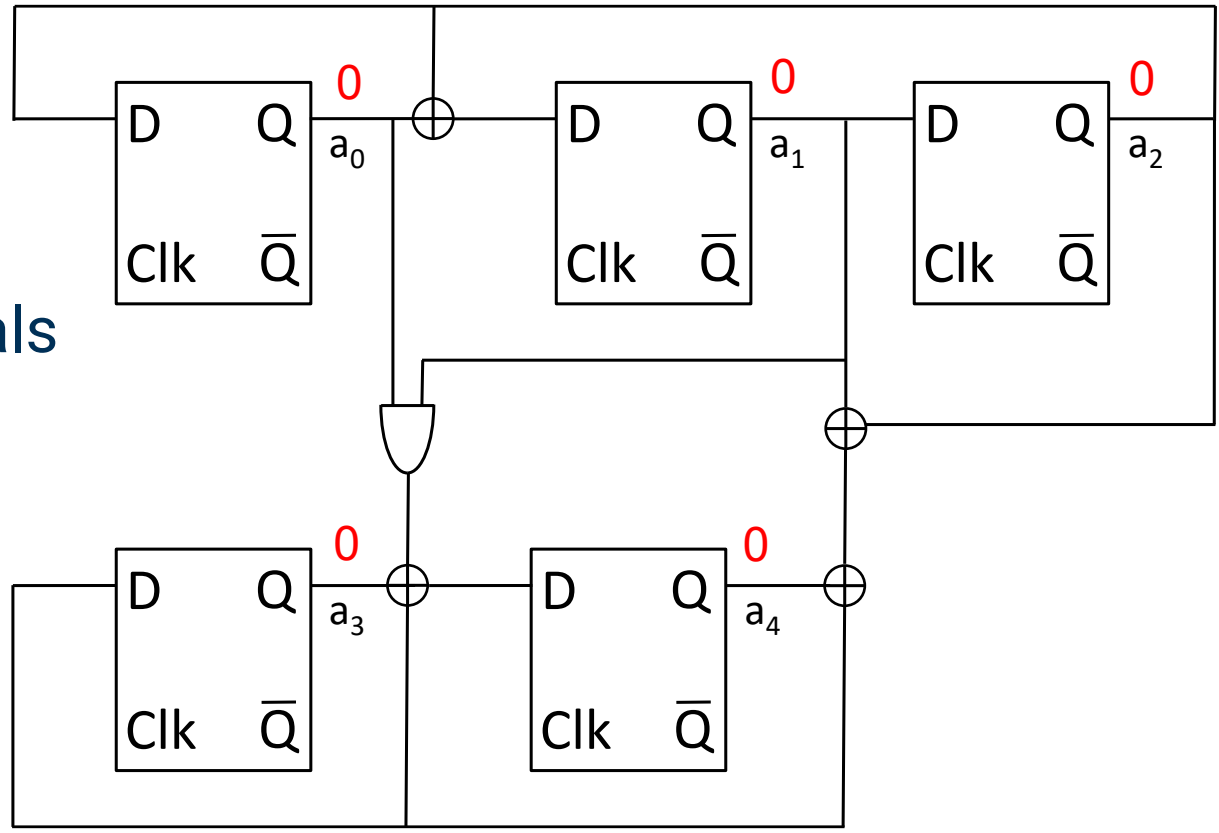
- Consider starting state $a_2,a_1,a_0$ = 001 and $a_4,a_3$ = 01; the result is a cycle of size 21
- Consider starting state $a_2,a_1,a_0$ = 000 and $a_4,a_3$ = 01; the result is a cycle of size 3
- Consider starting state $a_2,a_1,a_0$ = 001 and $a_4,a_3$ = 00; the result is a cycle of size 7

# Example 1




- Shown is a 3-bit LFSR and a 2-bit LFSR
- These two LFSRs with primitive polynomials were shown earlier
- The 2-bit LFSR is relabeled $a_4$, $a_5$
- The chaining function is $a_1$ XOR $a_2$ and $a_0$ AND $a_1$
- The next state functions for the 2-bit LFSR are now $a_4 = a_3$ XOR ($a_0$ AND $a_1$) and $a_3 = a_4$ XOR $a_1$ XOR $a_2$
- Consider starting state $a_2,a_1,a_0$ = 001 and $a_4,a_3$ = 01; the result is a cycle of size 21
- Consider starting state $a_2,a_1,a_0$ = 000 and $a_4,a_3$ = 01; the result is a cycle of size 3
- Consider starting state $a_2,a_1,a_0$ = 001 and $a_4,a_3$ = 00; the result is a cycle of size 7
- Consider starting state $a_2,a_1,a_0$ = 000 and $a_4,a_3$ = 00; result cycle size 1

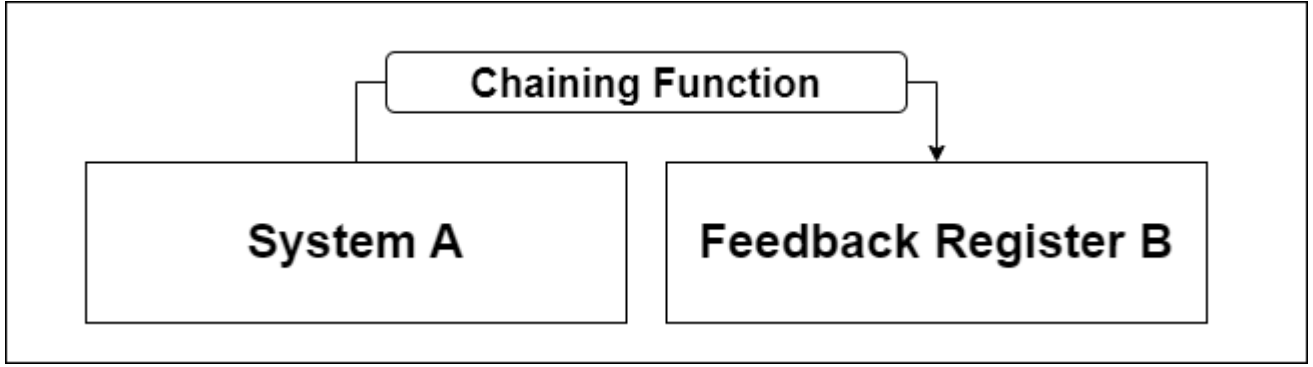



Figure 2(a) from [1]

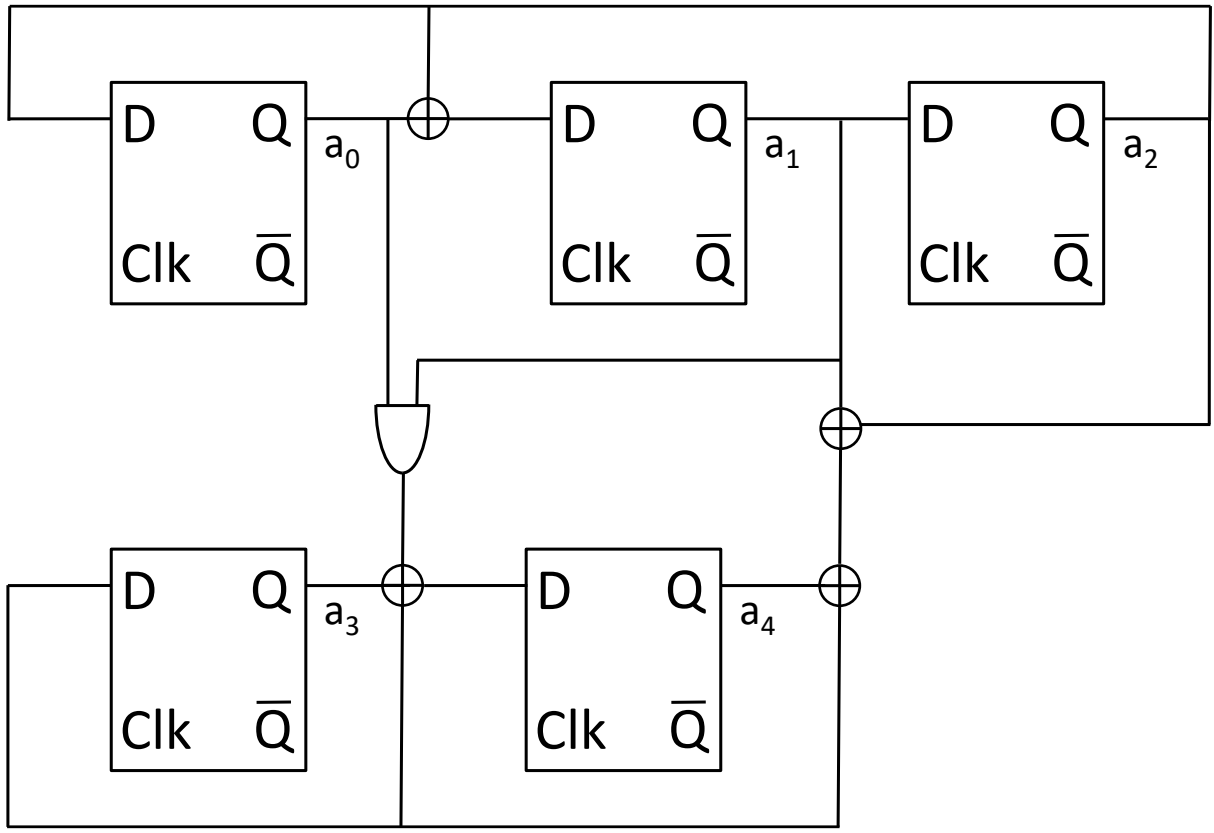
***THEOREM 1 (Chaining Period Theorem)*** *As shown in Figure 2(a), let A=(S,f) be a nonsingular system, and let B be an n-bit register with linear feedback represented by the matrix* $U \epsilon \mathbb{GL}(n,2)$*. Let the characteristic polynomial of U be irreducible, and let B have period m. Let* $C: S \to \mathbb{F}_2^n$ *be some chaining function from A to B, and let R denote the composite system formed by A and B via chaining function C. Then the cycle structure of R can be determined from the cycles of A and B as follows:*

*For any cycle of length p in A:*

1. *If* $m \mid p$ *and* $f_p = 1_{G_U}$ *then there are* $2^n$ *cycles of length p*
2. *If* $m \mid p$ *and* $f_p \neq 1_{G_U}$ *then there are* $2^{n-1}$ *cycles of length 2p*
3. *If* $m \nmid p$ *then there are* $\frac{2^n-1}{m}\gcd(m,p)$ *cycles of length* $\mathrm{lcm}(m,p)$ *and one cycle of length* $p$

# Example 1 (continued)

- The 3-bit LFSR has a cycle of size $p = 7$
- The 2-bit LFSR has a cycle of size $m = 3$
- Clearly, $m$ does not evenly divide $p$
- Therefore, according to the Chaining Period Theorem, corresponding to the cycle of size 7 in the 3-bit LFSR, the overall 5-bit register has a cycle of size 21 and a cycle of size 7




- More formally, $\mathrm{lcm}(m,p) = 21$, and $\frac{2^n-1}{m}\gcd(m,p) = \frac{2^2-1}{3}\gcd(3,7) = 1$, so for the cycle of size $p = 7$ in the 3-bit LFSR the overall 5-bit register has cycles of size $\mathrm{lcm}(m,p) = 21$ and $p = 7$
- Similarly, for the cycle of size 1 (in other words, $p = 1$) in the 3-bit LFSR, the overall 5-bit register has cycles of size $\mathrm{lcm}(m,1) = 3$ and $p = 1$

# Chaining Period Theorem Works for Nested Chaining

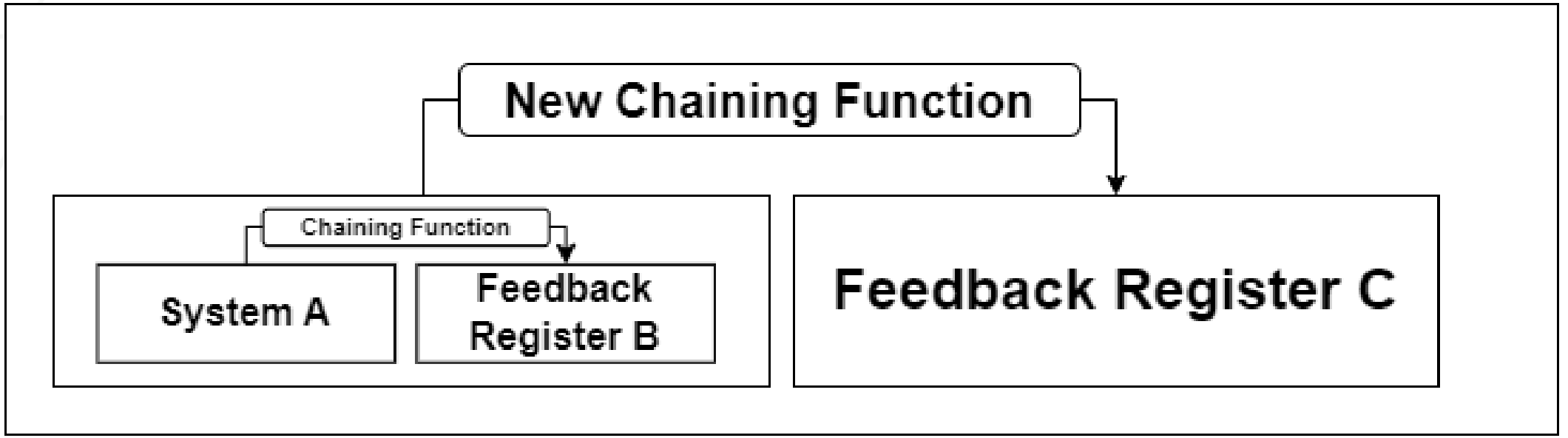



**Figure 2(b) from [1]**

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# Product Registers

- The Product Register (PR) [1] is a generalization of the Galois LFSR, allowing for the internal state to be permuted in a general way rather than restricting to a shift
- An $n$-bit Product Register is parametrized by:
  - A feedback polynomial $P(x)$, where $\deg\big(P(x)\big) = n$
  - An update polynomial $U(x)$, where $\deg(U(x)) \leq n - 1$
- The state of a PR updates according to:
  - $\boldsymbol{A[t+1] = U(x)A[t] \bmod P(x)}$
- From the above, it is evident that $U(x) = 0, 1$ result in degenerate behavior:
  - $U(x) = 0$ causes a PR to remain stuck in the all-zeros state
  - $U(x) = 1$ causes a PR to remain stuck in its initial state
- Thus, we omit $U(x) = 0, 1$ from the set of "valid" update polynomials for a PR
  - => An $n$-bit PR has $2^n - 2$ possible valid update polynomials
- We consider Galois LFSRs as a subset of PRs, namely the special case where $U(x) = x$

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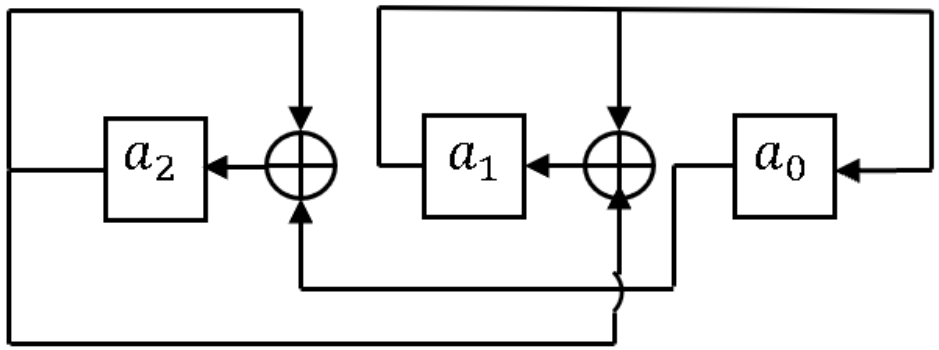
# Mersenne Product Registers

- A Mersenne Product Register (MPR) is a Product Register whose size is a Mersenne exponent
  - Ex: 2-bit PR, 3-bit PR, 5-bit PR are all MPRs
  - Larger MPRs: 61-, 89-, 127-bit
  - After $n = 127$, gaps between Mersenne exponents become much larger
- From Definition 20 of [1], an $n$-bit MPR achieves period $2^n - 1$ provided $P(x)$ is primitive and $U(x) \neq 0, 1$
  - For an MPR, $\deg(P(x))$ is prime; thus, any irreducible $P(x)$ is also primitive [12]




**A 3-bit MPR with $P(x) = x^3 + x + 1$ and $U(x) = x^2$**

# Mersenne Product Registers

- The set of possible states of an MPR always remains the same
- Changing $U(x)$ alters the order in which an MPR traverses its possible states when the MPR is seeded to the same initial state
- Example from [1]:
  - 3-bit MPR
  - Fixed primitive $\mathrm{P(x)} = \mathrm{x}^3 + \mathrm{x} + 1$ => MPR has full period of $2^3 - 1 = 7$
  - Fixed initial state: $001$
  - Vary $\mathrm{U(x)}$

| Update | $t=0$ | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $t=6$ |
|---|---|---|---|---|---|---|---|
| $x$ | 001 | 010 | 100 | 101 | 111 | 011 | 110 |
| $x+1$ | 001 | 011 | 101 | 010 | 110 | 111 | 100 |
| $x^2$ | 001 | 100 | 111 | 110 | 010 | 101 | 011 |
| $x^2+1$ | 001 | 101 | 110 | 100 | 011 | 010 | 111 |
| $x^2+x$ | 001 | 110 | 011 | 111 | 101 | 100 | 010 |
| $x^2+x+1$ | 001 | 111 | 010 | 011 | 100 | 110 | 101 |

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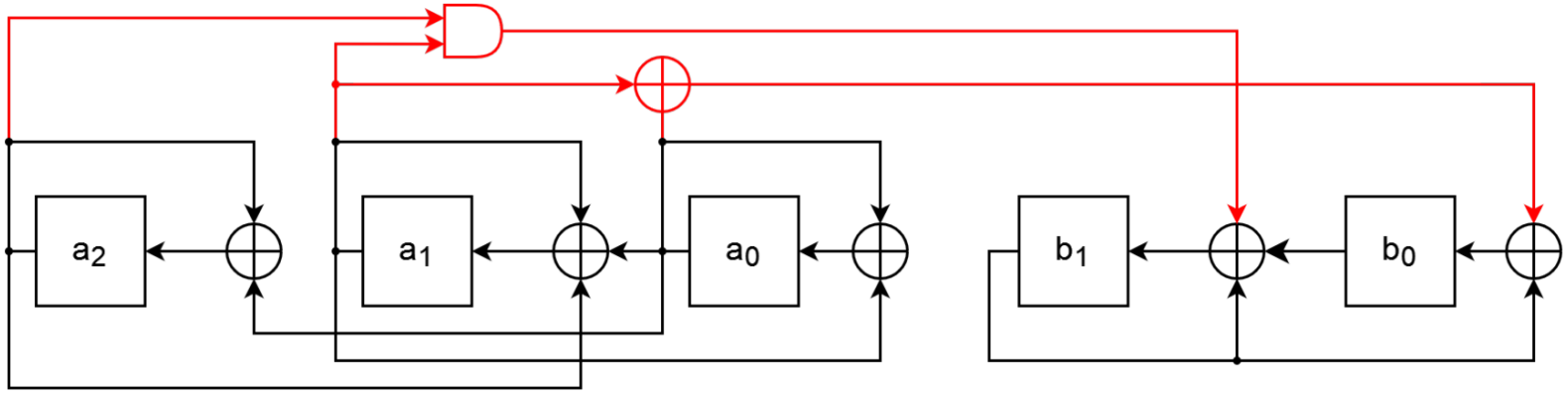
# Composite Mersenne Product Registers

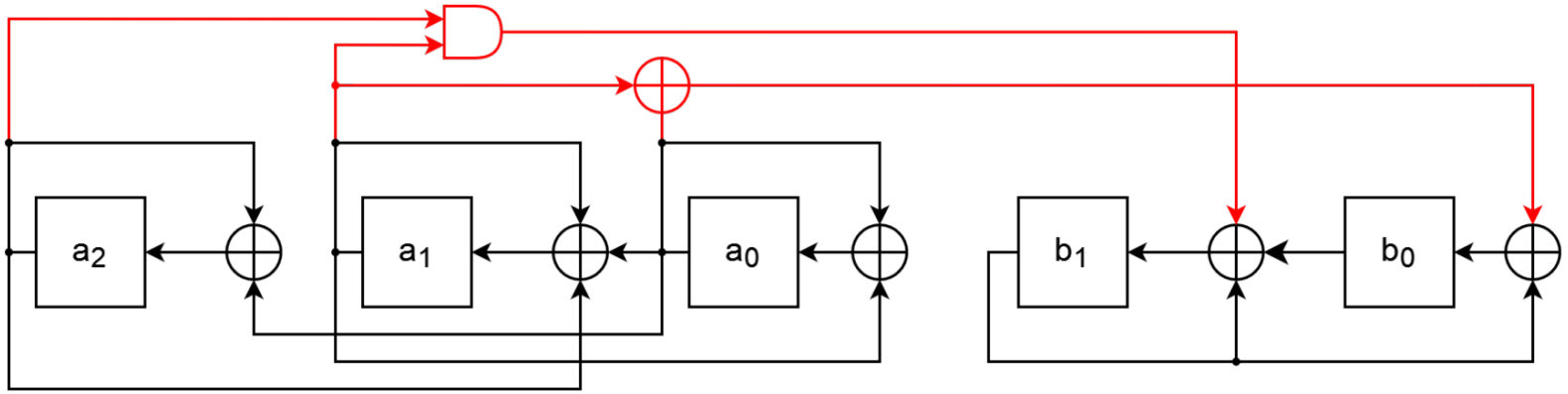
- On their own, MPRs are linear, meaning they cannot be used as a standalone cryptographic primitive
- It is possible to connect MPRs to form a larger nonlinear structure called a Composite Mersenne Product Register (CMPR)
- MPRs can be connected using **chaining functions**, or sets of Boolean functions that allow the state of an MPR to affect another MPR
  - The resulting structure is a CMPR
- CMPRs can be made nonlinear by using multiplicative terms (e.g., AND gates) in the chaining functions
- **Cryptographic Analogy:** MPRs provide a linear diffusion mechanism, whereas chaining functions provide nonlinear confusion




**A 5-bit CMPR formed by chaining a 3-bit MPR into a 2-bit MPR (Chaining function shown in red)**

# Composite Mersenne Product Registers

- In the 5-bit CMPR below, denote the state of the 3-bit MPR by $\mathrm{A}_1[\mathrm{t}]$, the state of the 2-bit MPR by $\mathrm{A}_2[\mathrm{t}]$, and the chaining function by $\mathrm{C}$
- Similarly, denote the update polynomial and primitive polynomial of the 2-bit MPR by $\mathrm{U}_2(\mathrm{x})$ and $\mathrm{P}_2(\mathrm{x})$, respectively
- Then, the state update of the 2-bit MPR becomes:
  - $\boldsymbol{A_2[t+1] = [U_2(x)A_2[t] \oplus C] \bmod P_2(x)}$
- In the example below, $C = a_2a_1 \oplus a_1 \oplus a_0$




**A 5-bit CMPR formed by chaining a 3-bit MPR into a 2-bit MPR (Chaining function shown in red)**

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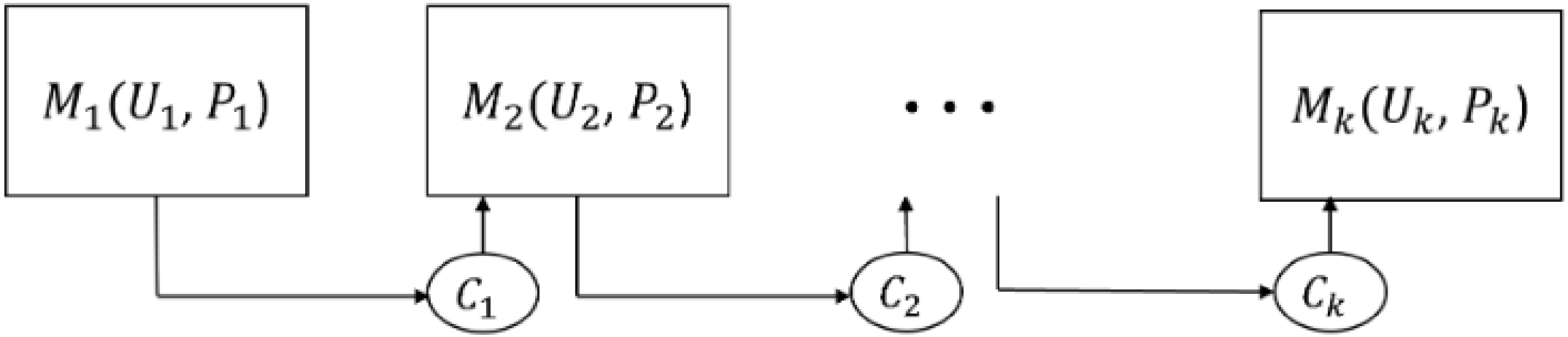
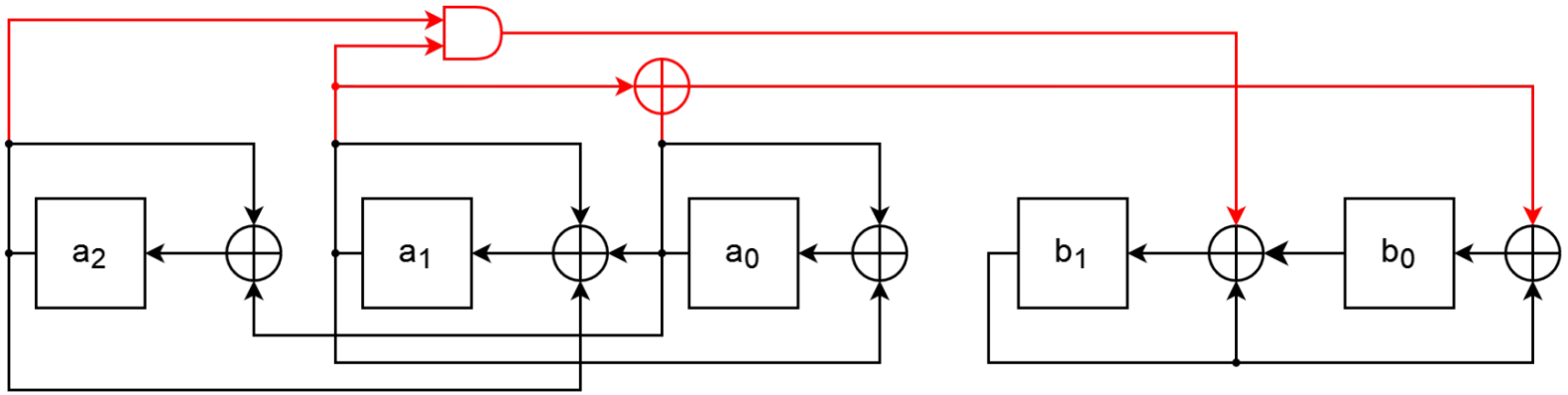
***THEOREM 2*** *Let R be a CMPR composed of k MPRs, denoted* $\{M_1, M_2, \ldots, M_k\}$*, with sizes* $S = \{m_1, m_2, \ldots, m_k\}$ *where each* $m_i$ *is a unique Mersenne exponent. Then R has a unique cycle of states for each subset* $S = \subseteq \{m_1, m_2, \ldots, m_k\}$*, with that cycle having a length equal to* $\prod_{m_i \epsilon S}(2^{m_i} - 1)$.

Proof by induction available in [1].



# CMPR Example for Theorem 2

- In the 5-bit CMPR below, the 3-bit MPR can be considered $M_1$ in Theorem 2 with size $m_1 = 3$
- Similarly, the 2-bit MPR can be considered $M_2$ in Theorem 2 with size $m_2 = 2$
- Then, the 5-bit CMPR has a cycle of size 21
  - $\prod_{m_i \epsilon S}(2^{m_i} - 1) = (2^3 - 1)(2^2 - 1) = 21$




**A 5-bit CMPR formed by chaining a 3-bit MPR into a 2-bit MPR (Chaining function shown in red)**





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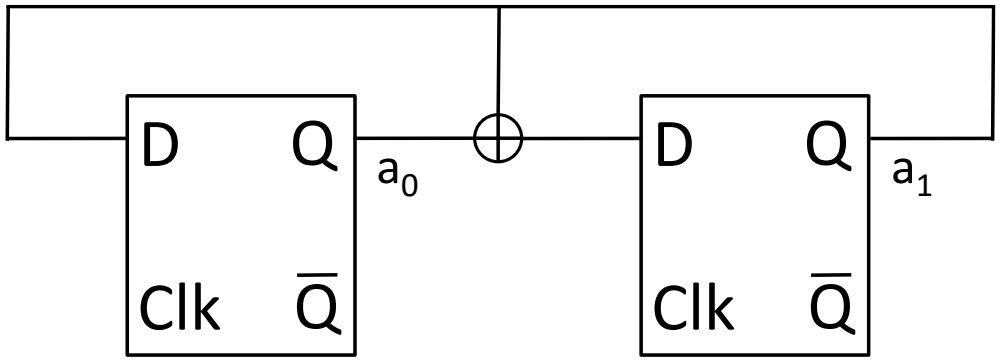
**DEFINITION 23** The Expected Period Ratio (EPR) of a CMPR of size *n* is given by $\frac{\mathbb{E}(X)}{2^n}$, where $\mathbb{E}(X)$ is the expected value of the period when the CMPR is initialized to a random state.

**DEFINITION 23** The Expected Period Ratio (EPR) of a CMPR of size *n* is given by $\frac{\mathbb{E}(X)}{2^n}$, where $\mathbb{E}(X)$ is the expected value of the period when the CMPR is initialized to a random state.




$$EPR = \frac{\mathbb{E}(X)}{2^n} = \frac{1}{2^2}\left(\frac{3}{4}3 + \frac{1}{4}\right) = \frac{10}{16} = 0.625$$

***THEOREM 3*** *Let C be a CMPR composed of k MPRs, denoted* $\{M_1, M_2, \dots, M_k\}$*, with sizes* $S = \{m_1, m_2, \dots, m_k\}$ *where each* $m_i$ *is a unique Mersenne exponent. Let n be the total number of bits in register C. The total number of states in C is then* $2^n = \prod_{m_i \epsilon S} 2^{m_i}$*. Let the initial state of C be chosen uniformly at random, and let the random variable* $X$ *denote the length of the cycle before this initial state repeats (the period of the register, started from this state). Then,*

$$\frac{\mathbb{E}(X)}{2^n} = \prod_{m_i \epsilon S} \left(1 - \frac{\left(2^{(m_i+1)} - 2\right)}{2^{2m_i}}\right)$$

Proof by induction available in [1].

# Some Examples of Theorem 3

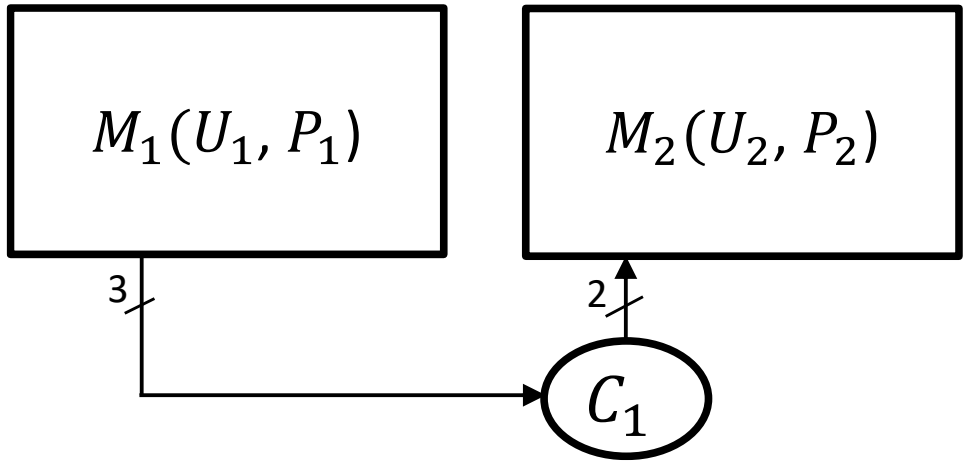
- MPR $M_1$ with size $m_1$

$$M_1(U_1, P_1)$$

$$\mathbb{E}(X) = \left(1 - \frac{\left(2^{(m_i+1)} - 2\right)}{2^{2m_i}}\right) = \frac{2^{2m_i} - 2^{(m_i+1)} + 2}{2^{2m_i}} = \frac{(2^{m_i} - 1)^2 + 1}{2^{2m_i}}$$

For $m_1 = 3$, $\frac{(2^{m_i}-1)^2+1}{2^{2m_i}} = \frac{(2^3-1)^2+1}{2^6} = \frac{50}{64} = 0.78125$

# Some Examples of Theorem 3 (continued)



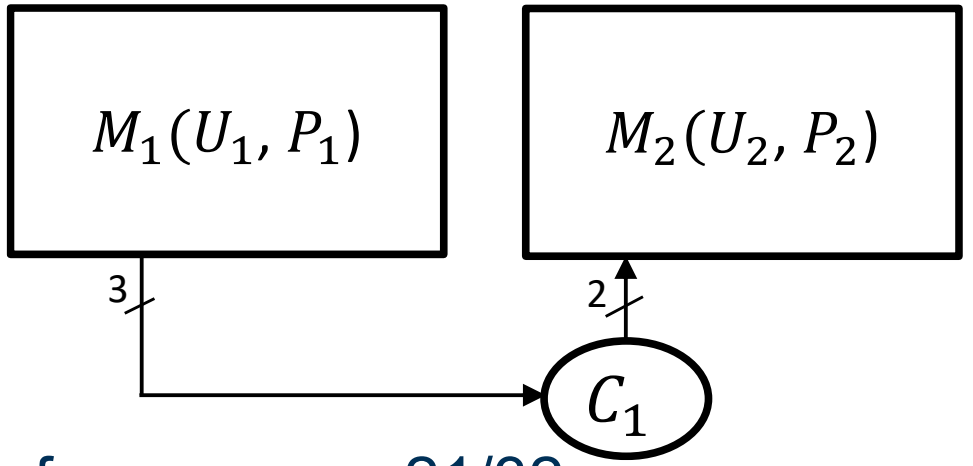

- CMPR: $M_1$ with size $m_1$=3 and $M_2$ with size $m_2$=2

For $m_2$ = 2, $\frac{(2^{m_i}-1)^2+1}{2^{2m_i}} = \frac{(2^2-1)^2+1}{2^4} = \frac{10}{16} = 0.625$

For $m_1$ = 3, $\frac{(2^{m_i}-1)^2+1}{2^{2m_i}} = \frac{(2^3-1)^2+1}{2^6} = \frac{50}{64} = 0.78125$

For this CMPR, $\mathrm{EPR} = (0.625 * 0.78125) = 0.48828125$

# Some Examples of Theorem 3 (continued)




- CMPR: $M_1$ with size $m_1$=3 and $M_2$ with size $m_2$=2

This CMPR has a cycle of size 21 with probably of occurrence 21/32
A second cycle of size 7 occurs with probably 7/32
A third cycle of size 3 occurs with probability 3/32
Finally, a cycle of size 1 exist with probability 1/32

$$\mathbb{E}(X) = 21\frac{21}{32} + 7\frac{7}{32} + 3\frac{3}{32} + \frac{1}{32} = \frac{500}{32}$$

$$\Rightarrow \frac{\mathbb{E}(X)}{32} = \frac{500}{1024} = 0.48828125$$

# Expected Period and Ratio Bounds

## Table 3: Lower EPR Bound

| Smallest MPR Size | Approximate EPR Lower Bound |
|---|---|
| 2 | 0.4514 |
| 3 | 0.7223 |
| 5 | 0.9246 |
| 7 | 0.9842 |
| 13 | 0.9997 |

## Table 4: Example Constructions

| CMPR Size | Mersenne Exponents | Expected Period | Expected Period Ratio (EPR) |
|---|---|---|---|
| 32 | 19, 13 | $4.2939 \times 10^{9}$ | 0.99975 |
| 64 | 61, 3 | $1.4412 \times 10^{19}$ | 0.78125 |
| 128 | 61, 31, 19, 17 | $3.4028 \times 10^{38}$ | 0.99998 |
| 256 | 127, 61, 31, 17, 13, 7 | $1.1397 \times 10^{77}$ | 0.98424 |

# Composite Mersenne Product Registers

- For a CMPR formed by MPRs $\{M_1, \ldots, M_k\}$, the size of the CMPR is given by $\sum_{i=1}^{k} \text{len}(M_i)$
- CMPRs also have a mathematically-proven **expected period** [1], which is the number of unique states the CMPR will, on average, cycle through when initialized to a uniformly-chosen initial state
- In this presentation, we assume the following conventions are followed when constructing a CMPR:
  - Each Mersenne exponent is used once
  - Chaining functions connect from larger MPRs to smaller MPRs
  - Feedback polynomials for all MPRs are irreducible (primitive) and $U(x) \neq 0, 1$
- Then, from Theorem 3 of [1], an $n$-bit CMPR has an expected period $\geq \mathbf{0.45 * 2^n \approx 2^{n-1}}$
- More generally, the expected period of an $n$-bit CMPR constructed from a set of Mersenne exponents $S$ is given by:
  - $\mathbf{[\sum_{m_i \in S}(1 - \frac{2^{m_i+1}-2}{2^{2m_i}})] * 2^n}$ **[1]**

- The expected period is independent of the chaining functions used to construct a CMPR
  - Nonetheless, it is advantageous to use **balanced Boolean equations** as chaining functions to ensure that the CMPR state evolution is not biased
  - In other words, the truth tables for each Boolean equation used as a chaining function should consist of equal numbers of 0's and 1's

***THEOREM 4*** *Let C(A[t]) be any periodic sequence of n-bit vectors in $\mathbb{F}_2^n$, and let its Z-transform be denoted C(z). Let B be an n-bit linear feedback register, with n prime and its characteristic polynomial primitive, and which updates according to Equation 1, with C(A[t]) as input on each cycle. Then the Z-transform of B[t] is that*

$$B(z) = (zI \oplus U)^{-1}(C(z) \oplus zB[0]) = \frac{1}{\chi_u(z)}\mathrm{Adj}(zI \oplus U)(C(z) \oplus zB[0]))$$

*where $\chi_u(z)$ is the characteristic polynomial of $U$, the matrix by which $B$ updates.*

# Theorem 4 Provides the Following

- Analysis of linear complexity of CMPRs using root expressions
- The ability to bound the linear complexity of CMPRs without the need to generate a long sequence
  - Instead, the $B(z)$ representation of Theorem 4 can be used to analyze how nonlinear expressions propagate through a CMPR
- Enables block-by-block propagation of root expressions, allowing CMPRs to be evaluated compositionally rather than through brute-force sequence generation

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